

Crystal-symmetry preserving Wannier states for fractional chern insulators

Chao-Ming Jian and Xiao-Liang Qi

Department of Physics, Stanford University, Stanford, CA 94305

Recently, many numerical evidences of fractional Chern insulator, i.e. the fractional quantum Hall states on lattices, are proposed when a Chern band is partially filled. Some trial wavefunctions of fractional Chern insulators can be obtained by mapping the fractional quantum Hall wavefunctions defined in the continuum onto the lattice through the Wannier state representation (Phys. Rev. Lett. **107**, 126803 (2011)) in which the single particle Landau orbits in the Landau levels are identified with the one dimensional Wannier states of the Chern bands with Chern number $C = 1$. However, this mapping generically breaks the lattice point group symmetry. In this paper, we discuss a general approach of modifying the mapping to accommodate the lattice rotational symmetry. The wavefunctions constructed through this modified mapping should serve as better trial wavefunctions to compare with the numerics and also as the basis for construction of lattice symmetry preserving pseudo-potential formalism for fractional Chern insulators. The focus of this paper shall be mainly on the C_4 rotational symmetry of square lattices. Similar analysis can be straightforwardly generalized to triangular or hexagonal lattices with C_6 symmetry. We also generalize the discussion to the lattice symmetry of fractional Chern insulators with high Chern number bands.

I. INTRODUCTION

Integer and fractional quantum Hall states differ from the conventional phase of matter characterized via symmetry breaking by their distinct topological properties. Under strong magnetic field, the 2D electron gas forms highly degenerate Landau Levels (LL) which carry non-trivial topological indices known as the TKNN invariants¹ or the Chern number. When a Landau level is fully occupied, the Chern number reveals itself as the quantized Hall conductance in integer quantum Hall effect (IQH effect). When it comes to a partial filling, the strong correlation between electrons arise from the huge degeneracy of LL (or the quenched kinetic energy), together with the its non-trivial topology, leads to the fractional quantum Hall states (FQH states) that support fractionalized excitations and non-trivial braiding statistics. In a work in 1988², F. D. M. Haldane proposed the first realization of a band insulator model with non-trivial Chern bands (i.e. bands with non-zero Chern number). A fully occupied Chern band also shows a non-zero Hall conductance, which is referred to as quantum anomalous Hall (QAH) insulator, or Chern insulator. This resemblance between the Chern bands and that LLs is rooted in their equivalence in the sense of topology. Now it is conceivable that, if the Chern band is flat or almost flat, fractional Chern insulators (FCI), which are FQH states without magnetic field, will emerge. Following this line of thought, lots of effort has been devoted into the search for FCI states^{3–19}.

For the FQH states, wavefunctions, the validity of which can be checked most straightforwardly through numerics, have been a major tool to study the underlying physics. However, the understanding of FQH states through wavefunctions such as the Laughlin wavefunction is not directly applicable to FCI states, since the former can be written as holomorphic functions and the latter is defined on a lattice. As proposed in Ref. 15, this

difficulty can be overcome through the mapping between the LLs and Chern bands that exploit their similarity in the Wannier state representation (WSR). In a Chern band, the single particle Wannier states that are localized on one direction and carry well-defined wave vectors on the other resemble the Landau orbits in the LLs. By directly identifying them in the Fock basis, all FQH wavefunctions written in the continuum can be mapped into FCI wavefunctions. In the WSR, there is a gauge choice in the definition of the Wannier states in the Chern bands. Different gauge choices, though capturing the same topological properties, lead to microscopically different FCI states through the mapping. Attempts have been made to gauge-fix the Wannier states for the comparison with numerics^{20–23}. However, these gauge fixing schemes generically render the lattice rotation symmetry broken in the mapping to the LLs unless the Berry curvature in the Brillouin zone is homogeneous²¹. An exception is the Coulomb gauge in Ref. 20 which preserves the four-fold lattice rotation symmetry, though it is not constructed for this purpose. This gauge choice will be further discussed later.

In this paper, we discuss the general procedure of constructing Wannier states that preserves crystal symmetry. Rather than using WSR, we equivalently identify the Bloch states of the LLs and Chern bands, similar to Ref. 20. We show that there is a series of gauge choices which leads to Bloch states in Chern bands that transform in the same fashion as that in the LLs. With these gauge choices, coherent states for the Chern bands¹⁵ preserving the lattice rotation symmetry can be constructed. This implies that the FCI states and their corresponding pseudo-potential Hamiltonians constructed through the mapping between the Bloch states in the LLs and the Chern bands are invariant under the lattice rotation. In the following, we will first introduce the Bloch state representation of the LLs and discuss its rotation properties. Then we will use the lattice Dirac model²⁴ and the

Checkerboard model⁴ as examples to demonstrate the construction of the rotation symmetry preserving gauge for the Chern bands with $C = 1$. In the end, we will discuss the case with higher Chern number.

II. GENERAL CRITERIA

In the original proposal¹⁵, a LL is mapped to a Chern band by identifying the Landau orbits with the Wannier stats. Equivalently, we can think of this mapping as an identification between Bloch states of the LL (that will be defined later) and those of the $C = 1$ Chern bands.²⁰ Although not explicit in this Bloch state representation, the LL does have a continuous $SO(2)$ rotation symmetry about the z axis which includes a lattice rotation symmetry as a subgroup. Thus, lattice rotation symmetric FCI states can be obtained from $SO(2)$ rotation symmetric FQH states in the continuum written in the Fock basis by mapping the LL to the Chern bands in the Bloch state representation and requiring that the Bloch states in the LL and the Chern bands share the same transformation properties under the lattice rotation. As the first step, we should define the Bloch states of LL and study their behavior under lattice rotation. In the rest of the paper, we will mainly focus on the C_4 rotation symmetry of square lattices.

In the Landau gauge, the Landau orbits can be written as

$$\psi_{p_y}(x, y) = \frac{1}{\sqrt{\sqrt{\pi} L_y l_b}} e^{ip_y y} e^{-\frac{1}{2l_b^2}(x - l_b^2 p_y)^2}, \quad (1)$$

where L_y is the size of the system in y direction, $l_b = \sqrt{\hbar/(eB)}$ is the cyclotron length and $p_y = 2\pi n/L_y$, $n \in \mathbb{Z}$ is the wavevector along y direction. As we can see, the Landau orbits are plane waves along the y direction but localized along the x direction. Now, we introduce a square lattice to the system with lattice constant a satisfying $a^2 = 2\pi l_b^2$. Since each unit cell contains one flux quantum, the translations along the two base vectors commute with each other, which allows us to define Bloch states as an alternative basis of the LL. Consider an $N \times N$ system (in terms of lattice constant a). The Bloch wavefunction can be written as

$$\psi_{\mathbf{k}}^{bl}(x, y) = \frac{1}{\sqrt{N}} \sum_{n \in \mathbb{Z}} e^{ik_x n a} \psi_{k_y + 2n\pi/a}(x, y), \quad (2)$$

where $\mathbf{k} = (k_x, k_y)$ is the Bloch momentum with $k_x, k_y \in [-\frac{\pi}{a}, \frac{\pi}{a}]$ and $k_{x,y} = 2\pi n_{x,y}/N$, $n_{x,y} \in \mathbb{Z}$. Note that

$$\psi_{\mathbf{k}}^{bl} = \psi_{\mathbf{k}+\mathbf{g}_x}^{bl} = e^{ik_x a} \psi_{\mathbf{k}+\mathbf{g}_y}^{bl}, \quad (3)$$

where $\mathbf{g}_{x,y}$ are the two base vectors of the reciprocal lattice. The Bloch wavefunction can be expressed in the standard form

$$\psi_{\mathbf{k}}^{bl}(x, y) = e^{ik_x x + ik_y y} u_{\mathbf{k}}(x, y), \quad (4)$$

with

$$u_{\mathbf{k}} = \frac{1}{\sqrt{\sqrt{\pi} N^2 a l_b}} \sum_n e^{ik_x (na-x) + i \frac{2n\pi y}{a}} e^{-\frac{(x - l_b^2 k_y - na)^2}{2l_b^2}}. \quad (5)$$

One can show straightforwardly that the Berry connection \mathbf{A} in the Brillouin zone (BZ) is given by

$$\mathbf{A} = -i \langle u_{\mathbf{k}} | \nabla_{\mathbf{k}} | u_{\mathbf{k}} \rangle = (-l_b^2 k_y, 0), \quad (6)$$

from which we see that the Chern number of the LL is $C = 1$ by integrating the Berry curvature across the whole BZ. From the definition of $u_{\mathbf{k}}(x, y)$ in Eq. 5, one can derive that

$$u_{\mathbf{k}}(y, -x) = e^{-ik_x k_y l_b^2} e^{-ixy/l_b^2} u_{R\mathbf{k}}(x, y), \quad (7)$$

where $R\mathbf{k} = (-k_y, k_x)$ is the Bloch momentum \mathbf{k} rotated by 90° . Eq. 7 describes how the Bloch wavefunctions transform under the C_4 rotation in real space. On the R.H.S. of Eq. 7, $u_{R\mathbf{k}}(x, y)$ is what would be normally expected for Bloch states under C_4 rotation. The factor e^{-ixy/l_b^2} generates a real space gauge transformation which follows a spatial C_4 rotation and restores the apparently broken rotation symmetry by the Landau gauge. The factor $e^{-ik_x k_y l_b^2}$ results from a specific gauge choice of the Bloch states in the BZ of the LL.

To extract the gauge condition of the Bloch states that is crucial for C_4 rotation symmetry of the LLs, we will introduce the coherent states ψ_{z_0} of the LL which form an overcomplete basis:

$$\psi_{z_0}(x, y) = \frac{1}{\sqrt{2\pi} l_b} e^{i \frac{xy}{2l_b^2}} e^{i \frac{x_0 y_0}{2l_b^2}} e^{-\frac{|z - z_0|^2 + (\bar{z} z_0 - z \bar{z}_0)}{4l_b^2}}, \quad (8)$$

where $z_0 = x_0 + iy_0$ is the center of mass of the coherent states. These states are localized in both directions. And they are invariant under the $SO(2)$ spatial rotation around its center up to a real space gauge transformation. We can re-express them in terms of the Bloch states:

$$\psi_{z_0} \propto \sum_n \sum_{k_{x,y} \in [-\frac{\pi}{a}, \frac{\pi}{a}]} e^{-\frac{y_0^2}{2l_b^2}} e^{-\frac{i^2}{2}(k_y + \frac{2n\pi}{a} - \frac{\bar{z}_0}{l_b^2})^2} e^{-ik_x n a} \psi_{\mathbf{k}}^{bl}. \quad (9)$$

This expression will serve as the general definition for coherent states on Chern bands when they are mapped to the LLs. We notice that it is the gauge condition Eq. 3 that guarantees the localization of ψ_{z_0} . One also can verify that, under the C_4 rotation, the coefficients in the second line conspire with the factor $e^{-ik_x k_y l_b^2}$ to guarantee the invariance of ψ_{z_0} under C_4 .

Thus, if we want to establish an identification between the Bloch states of the LL and those of the Chern bands without breaking the lattice rotation symmetry, the Bloch states of the Chern bands should satisfy the

same gauge condition: (1) Bloch states has the same periodicity as that in Eq. 3. (2) Under C_4 rotation, the Bloch wavefunction $u_{\mathbf{k}}$ satisfies:

$$u_{\mathbf{k}} = u_{\mathbf{k}+\mathbf{g}_x} = e^{ik_x a} u_{\mathbf{k}+\mathbf{g}_y}, \quad (10)$$

$$u_{R\mathbf{k}} = \hat{R}_{C_4} e^{ik_x k_y l_b^2} u_{\mathbf{k}}, \quad (11)$$

where \hat{R}_{C_4} denotes the symmetry transformation inside the unit cell, which is $(x, y) \rightarrow (-y, x)$ followed by a real space gauge transformation in the LL case. As a convention, we can always require that $\hat{R}_{C_4}^4 = 1$ and $u_{\mathbf{k}=0}$ is the eigenstate of \hat{R}_{C_4} with eigenvalue 1.

III. EXAMPLES

Lattice Dirac model - Now we use the lattice Dirac model as the first example to demonstrate how to preserve the C_4 symmetry by a gauge choice. The lattice Dirac model is given by the Hamiltonian $\hat{H} = \sum_{\mathbf{k}} c_{\mathbf{k}}^\dagger \mathcal{H}(\mathbf{k}) c_{\mathbf{k}}$, where $c_{\mathbf{k}}$ is a two-component fermion operator and

$$\mathcal{H}(\mathbf{k}) = \vec{\sigma} \cdot \mathbf{d}_{\mathbf{k}} \quad (12)$$

with

$$\mathbf{d}_{\mathbf{k}} = (\sin k_x, \sin k_y, M + \cos k_x + \cos k_y). \quad (13)$$

We have taken the lattice constant to be $a = 1$. M is a free parameter which is taken to be $0 < M < 2$ in order that the lower band has $C = 1$. This model is symmetric under a simultaneous C_4 rotation in both the real space and spin space:

$$\mathcal{H}(-k_y, k_x) = \hat{R}_{C_4} \mathcal{H}(k_x, k_y) \hat{R}_{C_4}^\dagger, \quad (14)$$

where $\hat{R}_{C_4} = e^{-i\frac{\pi}{4}} \begin{pmatrix} e^{-i\frac{\pi}{4}} & 0 \\ 0 & e^{i\frac{\pi}{4}} \end{pmatrix}$ is the spin rotation matrix times a $U(1)$ charge rotation which is chosen, for later convenience, to satisfy $\hat{R}_{C_4}^4 = 1$.

To gauge fix this Chern band, we can take the following two-step strategy. First, we write down the Bloch states in a gauge that respects the full periodicity of BZ and the C_4 rotation symmetry. Usually, this gauge will have singularities at the high symmetry points of the BZ. For the second step, we modify the gauge to satisfy the gauge condition Eq. 10 and Eq. 11. Since Eq. 11 only connects the Bloch states with momentums that are related to each other by C_4 rotation, we can pick any gauge in the "principle region" (PR) which is defined as a quarter of the BZ, $k_{x,y} \in [0, \pi/a]$ as shown in Fig. 1 (a), in this case. Then we use Eq. 11 to generate the gauge on other parts of the BZ. The requirement of self-consistency and the consistency with gauge condition Eq. 10 imposes only a boundary condition to the gauge choice in the PR.

Now, for the lattice Dirac model, we first write down the Bloch states in the lower band:

$$|k_x, k_y\rangle = \begin{pmatrix} \sin \frac{\theta_{\mathbf{k}}}{2} e^{-i\varphi_{\mathbf{k}}} \\ -\cos \frac{\theta_{\mathbf{k}}}{2} \end{pmatrix}, \quad (15)$$

where $\cos \theta_{\mathbf{k}} = d_{\mathbf{k}}^z / |\mathbf{d}_{\mathbf{k}}|$ and $\varphi_{\mathbf{k}} = \arg(d_{\mathbf{k}}^x + id_{\mathbf{k}}^y)$. This gauge satisfies

$$\begin{aligned} |k_x, k_y\rangle &= |k_x + 2\pi, k_y\rangle = |k_x, k_y + 2\pi\rangle, \\ | -k_y, k_x\rangle &= \hat{R}_{C_4} |k_x, k_y\rangle. \end{aligned} \quad (16)$$

It is easily shown that $|k_x, k_y\rangle$ is singular only at $\mathbf{k} = (\pi, \pi)$. For step two, we denote the Bloch states in the Dirac model which maps to those of the LL with the C_4 symmetry preserved as $|k_x, k_y\rangle = e^{i\phi(k_x, k_y)} |k_x, k_y\rangle$. The gauge conditions Eq. 10 and Eq. 11 then requires that

$$\begin{aligned} e^{i\phi(-k_y, k_x)} &= e^{i\frac{k_x k_y}{2\pi}} e^{i\phi(k_x, k_y)}, \\ e^{i\phi(k_x, k_y)} &= e^{i\phi(k_x + 2\pi, k_y)}. \end{aligned} \quad (17)$$

Notice that the "twisted" periodicity gauge condition along the k_y direction in Eq. 10 can be generated by these two equations. Therefore, the phase factor $e^{i\phi(k_x, k_y)}$ in the BZ is completely defined by its value in the PR with the following consistency condition:

$$\begin{aligned} \phi(k, 0) &= \phi(0, k_x), \\ \phi(k, \pi) + \frac{k}{2} &= \phi(-\pi, k) = \phi(\pi, k), \end{aligned} \quad (18)$$

where $k \in [0, \pi]$. Notice that the second equation leads to a singularity of $\phi(\mathbf{k})$ on the boundary of the PR, since

$$\lim_{k \rightarrow \pi} \phi(k, \pi) + \pi/2 = \lim_{k \rightarrow \pi} \phi(\pi, k). \quad (19)$$

This singularity correctly cancels that of $|k_x, k_y\rangle$ at $\mathbf{k}_M \equiv (\pi, \pi)$. All function $\phi(\mathbf{k})$ in PR that satisfies the gauge condition Eq. 18 and that are smooth only except at k_M can be consistently extended to the whole BZ by Eq. 17. As an example, one solution is given by:

$$\phi(\mathbf{k}) = \begin{cases} \left(1 - \frac{|\mathbf{k} - \mathbf{k}_M|}{\pi}\right) \theta'_{\mathbf{k}}, & \text{for } \frac{|\mathbf{k} - \mathbf{k}_M|}{\pi} < 1 \\ 0, & \text{elsewhere in PR} \end{cases}, \quad (20)$$

where $\theta'_{\mathbf{k}} = \arg(k_x + ik_y - (1+i)\pi) - \pi$. Now we have obtained $|k_x, k_y\rangle$ which corresponds to $u_{\mathbf{k}}$ in the LL. With this identification, coherent states for the Chern band in the lattice Dirac model can be defined through Eq. 9, and the C_4 rotation properties will be inherited from the LL.

At this stage, it seems that the cancelation of singularities between $\phi(\mathbf{k})$ and $|k_x, k_y\rangle$ is quite accidental. However, it is shown in Ref. 25 that the symmetry enforces constraints on the Chern number. And we will show that the Chern number dictates the singularity and leads to this cancelation. In the first step, we have chosen a gauge that satisfies Eq. 16 and usually leads to singularities at the high symmetry points: the Γ point with $\mathbf{k}_\Gamma = (0, 0)$, the X point with $\mathbf{k}_X = (\pi, 0)$, the Y point with $\mathbf{k}_Y = (0, \pi)$ and the M point with $\mathbf{k}_M = (\pi, \pi)$. The singularities at these points are $U(1)$ monodromy which can be determined by the eigenvalues of \hat{R}_{C_4} at these points. At the Γ point, by definition,

$$\lim_{\epsilon \rightarrow 0} |\mathbf{k}_\Gamma + \epsilon \mathbf{g}_y\rangle = \hat{R}_{C_4} |\mathbf{k}_\Gamma + \epsilon \mathbf{g}_x\rangle. \quad (21)$$

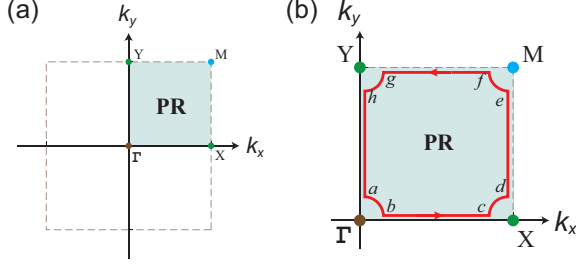


FIG. 1. (a) The BZ of square lattice with $k_{x,y} \in [-\pi, \pi]$ is indicated by the dash line. The blue area is the PR with its 4 corners being high symmetry points: Γ , X , Y and M . (b) The red line is a contour that follows the boundary of the PR. \widehat{ab} , \widehat{cd} , \widehat{ef} and \widehat{gh} are quarter circles with infinitesimal radii that centered around the 4 corners of the PR.

Meanwhile, $|\mathbf{k}_\Gamma\rangle$ must be an eigenstate of the \hat{R}_{C_4} with its eigenvalue denoted as ξ_Γ . Thus, we have

$$|\mathbf{k}_\Gamma + \epsilon \mathbf{g}_y\rangle \sim \xi_\Gamma |\mathbf{k}_\Gamma + \epsilon \mathbf{g}_y\rangle, \quad \epsilon \rightarrow 0. \quad (22)$$

Thus, the $U(1)$ phase singularity along the infinitesimal quarter circle \widehat{ab} is ξ_Γ^{-1} , and so is the Berry connection integral along the path \widehat{ab} in this gauge. Parallel analysis can also be done for M point relating the inverse of the eigenvalue of \hat{R}_{C_4} at the M point denoted as ξ_M with the $U(1)$ phase singularity and also the Berry connection integral along the path \widehat{ef} . For X and Y points, they are only invariant under $\hat{R}_{C_4}^2$. We denote the eigenvalue of $\hat{R}_{C_4}^2$ at X as η_X which equals that at the Y point. Through similar reasoning, the η_X^{-1} equals the total $U(1)$ phase singularity and the Berry connection integral along the path \widehat{cd} and \widehat{gh} . Now, we consider the total Berry connection integral along the red contour in Fig. 1 (b) which should be, when raised to the exponent, $e^{i\frac{2\pi C}{4}}$, namely a quarter of the total Berry curvature (on the exponent). Due to the gauge condition Eq. 16, the Berry connection integral along \widehat{ha} cancels that along \widehat{bc} . Same cancelation happens for \widehat{de} and \widehat{fg} . Thus, the only contribution to the integral comes from the four infinitesimal quarter circles:

$$e^{i\frac{2\pi C}{4}} = \xi_\Gamma^{-1} \xi_M^{-1} \eta_X^{-1}. \quad (23)$$

This result is also obtained in Ref. 25. As explained above, $\xi_\Gamma^{-1} \xi_M^{-1} \eta_X^{-1}$ also characterized the total $U(1)$ phase singularity along the four quarter circles at the 4 corners of the PR. For $C = 1$, comparing this total singularity of the gauge $|k_x, k_y\rangle$ with singularity of the phase factor $e^{i\phi(\mathbf{k})}$ required by Eq. 18, we see that they cancel each other.

Checkerboard model - As a second example system where C_4 symmetry acts differently, we consider the Checkerboard model that produces a flat Chern band⁴:

$$\mathcal{H}(\mathbf{k}) = \epsilon_{\mathbf{k}} I + h_{\mathbf{k}}^x \sigma_x + h_{\mathbf{k}}^y \sigma_y + h_{\mathbf{k}}^z \sigma_z, \quad (24)$$

where

$$\mathbf{h}_{\mathbf{k}} = \left(\cos \frac{k_x}{2} \cos \frac{k_y}{2}, -\sin \frac{k_x}{2} \sin \frac{k_y}{2}, \frac{\cos k_x - \cos k_y}{2\sqrt{2} + 2} \right) \quad (25)$$

and $\epsilon_{\mathbf{k}} = \frac{1}{\sqrt{2}+2} \cos k_x \cos k_y$ that does not affect the Bloch states. The BZ is defined as $k_x, k_y \in [-\pi, \pi]$ with the gauge condition

$$\sigma_z \mathcal{H}(k_x, k_y) \sigma_z = \mathcal{H}(k_x + 2\pi, k_y) = \mathcal{H}(k_x, k_y + 2\pi). \quad (26)$$

The C_4 symmetry and gauge property of this model is given by

$$\sigma_x \mathcal{H}(-k_y, k_x) \sigma_x = \mathcal{H}(k_x, k_y). \quad (27)$$

With the reference of the gauge condition Eq. 10 and Eq. 11, we need to find the Bloch states in a gauge such that:

$$\begin{aligned} |k_x + 2\pi, k_y\rangle &= \sigma_z |k_x, k_y\rangle \\ |-k_y, k_x\rangle &= -\sigma_x e^{i\frac{k_x k_y}{2}} |k_x, k_y\rangle. \end{aligned} \quad (28)$$

Following the two-step analysis described in the lattice Dirac model case, we obtain the general solution:

$$|k_x, k_y\rangle = e^{i\phi(\mathbf{k})} \begin{pmatrix} -\sin \frac{\theta_{\mathbf{k}}}{2} \\ \cos \frac{\theta_{\mathbf{k}}}{2} e^{i\varphi_{\mathbf{k}}} \end{pmatrix}, \quad (29)$$

where $\cos \theta_{\mathbf{k}} = h_{\mathbf{k}}^z / |\mathbf{h}_{\mathbf{k}}|$, $\varphi_{\mathbf{k}} = \arg(h_{\mathbf{k}}^x + ih_{\mathbf{k}}^y)$ and $e^{i\phi(\mathbf{k})}$ is a $U(1)$ phase that satisfies the following condition only on the boundary of the PR:

$$\begin{aligned} \phi(k, 0) &= \phi(0, k) \\ \phi(\pi, k) &= \frac{k}{2} - \frac{\pi}{2} + \phi(k, \pi), \end{aligned} \quad (30)$$

where $k \in [0, \pi]$. As expected, the singularity of the $\phi(\mathbf{k})$ around the boundary of PR can be set to cancel the singularity of the two-component spinor in Eq. 29. Now C_4 symmetric coherent states can be constructed via replacing $\psi_{\mathbf{k}}^{bl}$ in Eq. 9 with the Bloch states in Eq. 29. The wavefunction of a coherent state that centers at the origin is depicted in Fig. 2 (b).

In the end of this section, let's discuss why the Coulomb gauge in Ref. 20 preserves the C_4 symmetry. In the Coulomb gauge, the gauge potential $\mathbf{A}_{\mathbf{k}}$ is the sum of two parts $\mathbf{A} = \mathbf{A}_0 + \delta\mathbf{A}$ with \mathbf{A}_0 identical to the gauge potential in LL, and the correction $\delta\mathbf{A}$ with zero Chern number. Once gauge fixed by the Coulomb gauge condition, $\delta\mathbf{A}$ is single-valued in the BZ and transforms trivially under C_4 rotation. Therefore the C_4 rotation property of the Coulomb gauge is completely determined by \mathbf{A}_0 and thus is identical to the LL. Therefore the Coulomb gauge preserves the C_4 rotation symmetry in the mapping to the LL.

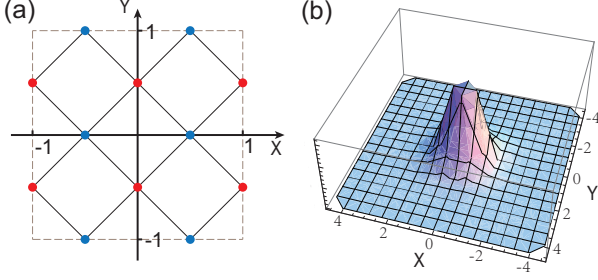


FIG. 2. (a) On the checkerboard lattice, the red and blue sites are the upper and lower components of the Bloch state. The coordinates are given in the unit of lattice constant. (b) The wavefunction of a coherent state that centers at the origin is symmetric under C_4 rotation.

IV. GENERALIZATION TO HIGHER CHERN NUMBER SYSTEMS

So far we have demonstrated that, for $C = 1$ Chern bands, we can always find the appropriate gauge such that the C_4 symmetry is preserved when the Chern band is mapped to the LL. Now we generalize the discussion to the case of Chern bands with Chern number $C > 1$. It has been shown in Ref. 16 that a Chern band with Chern number C can be mapped into C layers of Chern number 1 bands by WSR (given the condition that the system size is commensurate with C along the direction in which the Wannier states are localized). An alternative approach has been developed in Ref. 20. Since this mapping generically breaks the rotation symmetry, the existence of a rotationally symmetric scheme requires further inspection. In the following we will focus on the C_4 symmetry of FCI's in a $C = 2$ Chern band.

As is discussed in Ref. 16, in general, one-dimensional Wannier states of the $C = 2$ band can be defined by one-dimensional Fourier transform of the Bloch states $|\mathbf{k}\rangle$ along a reciprocal vector direction $n\hat{x} + m\hat{y}$. For each pair of mutually prime integers (n, m) , a set of Wannier states are defined, which defines a one-to-one mapping between bilayer FQH states and the FCI in this $C = 2$ band. If n is odd, the resulting FCI states is sensitive to the translation along x direction, such that an x -dislocation with Burgers vector $\mathbf{b} = \hat{x}$ will generically become a non-Abelian defect. Similarly, if m is odd the y -dislocation with Burgers vector $\mathbf{b} = \hat{y}$ becomes a non-Abelian defect. Therefore the topological properties of the FCI state necessarily break C_4 symmetry unless n and m are both odd. In the following we will study the C_4 symmetry of the simplest case $(n, m) = (1, 1)$ and discuss the more general cases later.

The Wannier states for $(n, m) = (1, 1)$ are defined as

$$|W_n(k_y)\rangle = \frac{1}{\sqrt{L}} \sum_{k_+ \in [0, 2\pi)} e^{ik_+ n} |k_+, k_+ + k_y\rangle \quad (31)$$

with $L = L_x = L_y$ the linear size of the system along both directions, and $|k_+, k_y\rangle$ the Bloch state of the $C = 2$

band. k_+ denotes the momentum along the diagonal direction and k_y denotes the momentum along y direction at $k_+ = 0$. The Bloch state can have different gauge choices and the goal of the following discussion is to find the conditions the C_4 symmetry impose to the gauge choice. For $C = 2$ bands, the Wannier states at even and odd sites form two families which are not connected by adiabatically changing k_y . Each family is topologically equivalent to a $C = 1$ band, or a Landau level. Since the FCI state constructed by WSR are bilayer FQH states with each family mapped to a layer of Landau level, the C_4 symmetry will be preserved if it is preserved for each layer. The Bloch states of each family can be obtained by an inverse Fourier transformation to each family of Wannier states in Eq. 31:

$$\begin{aligned} |k_+, k_+ + k_y\rangle_1 &= \frac{1}{\sqrt{L/2}} \sum_n |W_{2n-1, k_y}\rangle e^{-2ink_+} \\ |k_+, k_+ + k_y\rangle_2 &= \frac{1}{\sqrt{L/2}} \sum_n |W_{2n, k_y}\rangle e^{-2ink_+}. \end{aligned} \quad (32)$$

Here $k_+ \in [0, \pi)$ and $k_y \in [0, 2\pi)$ defines the reduced Brillouin zone. Combining Eq. 31 and 32, we obtain the following simple relation between the Bloch states of the $C = 2$ states and that of the effective $C = 1$ bands:

$$\begin{aligned} |k_+, k_+ + k_y\rangle_1 &= \frac{e^{-ik_+}}{\sqrt{2}} (|k_+, k_+ + k_y\rangle - |k_+ + \pi, k_+ + k_y + \pi\rangle) \\ |k_+, k_+ + k_y\rangle_2 &= \frac{1}{\sqrt{2}} (|k_+, k_+ + k_y\rangle + |k_+ + \pi, k_+ + k_y + \pi\rangle) \end{aligned}$$

or equivalently, in the conventional k_x, k_y basis,

$$\begin{aligned} |\mathbf{k}\rangle_1 &= \frac{1}{\sqrt{2}} (|\mathbf{k}\rangle - |\mathbf{k} + (\pi, \pi)\rangle) e^{-ik_x} \\ |\mathbf{k}\rangle_2 &= \frac{1}{\sqrt{2}} (|\mathbf{k}\rangle + |\mathbf{k} + (\pi, \pi)\rangle). \end{aligned} \quad (33)$$

To preserve C_4 symmetry in the two $C = 1$ bands, the Bloch states $|\mathbf{k}\rangle_{1,2}$ need to satisfy the C_4 invariance conditions (10) and (11). Eq. 33 together with Eq. 10 and 11 determines the general C_4 symmetry condition for the $C = 2$ FCI states constructed in the $(1, 1)$ WSR.

For more generic (n, m) with n, m odd, the discussion is exactly parallel and the reduced bands $|\mathbf{k}\rangle_{1,2}$ are determined by Eq. 33 with the momentum $Q = (\pi, \pi)$ replaced by $Q' = (n\pi, m\pi)$, and the factor e^{-ik_x} replaced by $e^{-i(tk_x - sk_y)}$ with s, t two integers satisfying $\det \begin{pmatrix} n & m \\ s & t \end{pmatrix} = 1$. For n, m odd, Q' and Q are equal modulo a reciprocal lattice vector. Therefore the effective $C = 1$ bands for all (n, m) odd are equivalent up to a phase factor.

In particular, the C_4 invariance condition imposes some constraints on the rotation eigenvalues of the $C = 2$ band at the high symmetry points. The original and reduced BZ's are shown in Fig. 3. In the same way as the $C = 1$ band discussed in the last section, we denote the

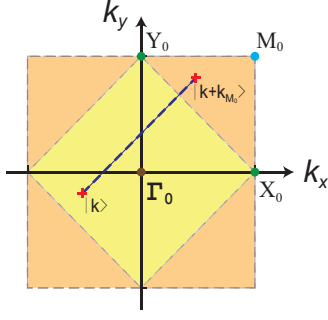


FIG. 3. A Chern number $C = 2$ band is mapped into two subbands with $C = 1$ by folding the BZ, namely taking a superposition of the states $|\mathbf{k}\rangle$ with $|\mathbf{k} + \mathbf{k}_{M_0}\rangle$.

R_{C_4} eigenvalue of the original $C = 2$ band at high symmetry points Γ_0, M_0 as $\xi_{\Gamma_0}, \xi_{M_0}$, and the $R_{C_4}^2$ eigenvalue at X_0 point as η_{X_0} . At the Γ points of the reduced BZ, the $C = 1$ Bloch states $|\mathbf{k}\rangle_{1,2}$ are superpositions of $|(0, 0)\rangle$ and $|\pi, \pi\rangle$ in the original BZ. Therefore the $C = 1$ bands at Γ point can only be C_4 invariant if $\xi_{\Gamma_0} = \xi_{M_0}$. The \hat{R}_{C_4} eigenvalue of the two $C = 1$ bands at the Γ point satisfies $\xi_{\Gamma_{1,2}} = \xi_{\Gamma_0} = \xi_{M_0}$. At the M points of the subbands, the Bloch states are the superpositions of $|\mathbf{k}_{X_0}\rangle_0$ and $|\mathbf{k}_{Y_0}\rangle_0$ which are both eigenstates of $\hat{R}_{C_4}^2$ with eigenvalue η_{X_0} . Hence, the eigenvalue of \hat{R}_{C_4} at $M_{1,2}$ should satisfy $\xi_{M_{1,2}}^2 = \eta_{X_0}$. At the X points of the subbands, the Bloch states are the superposition of $|\mathbf{k} = (-\frac{\pi}{2}, -\frac{\pi}{2})\rangle$ and $|\mathbf{k} = (\frac{\pi}{2}, \frac{\pi}{2})\rangle$. Since $\hat{R}_{C_4}^4 = 1$ by definition, the eigenvalue of $\hat{R}_{C_4}^4$ at $X_{1,2}$ should be $\eta_{X_{1,2}}^2 = 1$. By construction the two bands $|\mathbf{k}\rangle_{1,2}$ each have Chern number $C = 1$, so that we obtain based on Eq. 23 that $\xi_{\Gamma_0}^{-1}\xi_{M_0}^{-1}\eta_{X_0}^{-1} = -1$ and $\xi_{\Gamma_j}^{-1}\xi_{M_j}^{-1}\eta_{X_j}^{-1} = i$ with $j = 1, 2$. To summarize, the C_4 symmetry requires the following conditions to the rotation eigenvalues of the original band and the effective $C = 1$ bands:

$$\begin{aligned} \xi_{\Gamma_0}^{-1}\xi_{M_0}^{-1}\eta_{X_0}^{-1} &= -1, \\ \xi_{\Gamma_1}^{-1}\xi_{M_1}^{-1}\eta_{X_1}^{-1} &= \xi_{\Gamma_2}^{-1}\xi_{M_2}^{-1}\eta_{X_2}^{-1} = i, \\ \xi_{M_1}^2 &= \xi_{M_2}^2 = \eta_{X_0}, \quad \eta_{X_1}^2 = \eta_{X_2}^2 = 1, \\ \xi_{\Gamma_1} &= \xi_{\Gamma_2} = \xi_{\Gamma_0} = \xi_{M_0}. \end{aligned} \quad (34)$$

Since $\xi_{\Gamma_0} = \xi_{M_0}$ is required, it indicates that there are some $C = 2$ band which can never support any FCI state with C_4 symmetry, at least not one constructed by WSR.

For example, consider the model:

$$\begin{aligned} \mathcal{H}(k_x, k_y) &= (\sin^2 k_x - \sin^2 k_y)\sigma_x + 2 \sin k_x \sin k_y \sigma_y \\ &\quad + (1 + \cos k_x + \cos k_y)\sigma_z \end{aligned} \quad (35)$$

with $\hat{R}_{C_4} = \sigma_z$. The lower energy band has $\xi_{\Gamma_0} = -1$, $\xi_{M_0} = 1$, so that no C_4 invariant gauge choice can be found.

In summary, for a $C = 2$ Chern band on the square lattice, a scheme of mapping it into two $C = 1$ bands while preserving the C_4 rotation symmetry does NOT always exist. A possible obstruction can be detected by calculating the eigenvalue of the C_4 operation at high symmetry points in the BZ. The question remains: what are the conditions that guarantee a C_4 symmetry preserving splitting scheme of a Chern band with a generic Chern number C ? It is very likely that a generic Chern number will require an enlargement of unit cell that is incompatible with the rotation symmetry and, thus, will forbid a symmetry preserving splitting of the Chern band. This may be a reason to believe that the topological nematic state is generic.

V. CONCLUSION AND DISCUSSIONS

In conclusion, we propose a scheme to select the gauge of the Bloch states of $C = 1$ Chern bands so that the mapping from LL to the Chern bands respects the C_4 rotation symmetry of the square lattice. Consequently, FCI states obtained through this mapping are invariant under the C_4 rotation, and thus, are better trial wavefunctions to be compared with the numerics. Also, through the C_4 invariant coherent states, a C_4 symmetric pseudo-potential formalism can be constructed. Our discussion mainly focuses on the $C = 1$ Chern bands with C_4 symmetry on square lattices. Similar construction can be done for those with C_6 symmetry on triangular or hexagonal lattices. The discussion is generalized to higher Chern number case. A $C = 2$ band can be reduced to two $C = 1$ bands by WSR. The C_4 symmetry property of the $C = 2$ band can be determined by that of the two reduced $C = 1$ bands. C_4 symmetry imposes some constraints on the direction of the Wannier states in WSR, and also some constraints on the rotation eigenvalues at the high symmetry points. The conditions that guarantee a rotationally symmetric splitting scheme for a Chern band with an arbitrary Chern number C needs further investigation.

Acknowledgements - This work is supported by the David and Lucile Packard Foundation.

¹ D. J. Thouless, M. Kohmoto, M. P. Nightingale, and M. den Nijs, Phys. Rev. Lett. **49**, 405 (1982).

² F. D. M. Haldane, Phys. Rev. Lett. **61**, 2015 (1988).

³ E. Tang, J.-W. Mei, and X.-G. Wen, Phys. Rev. Lett. **106**, 236802 (2011).

⁴ K. Sun, Z. Gu, H. Katsura, and S. Das Sarma,

- Phys. Rev. Lett. **106**, 236803 (2011).
- ⁵ T. Neupert, L. Santos, C. Chamon, and C. Mudry, Phys. Rev. Lett. **106**, 236804 (2011).
 - ⁶ D. N. Sheng, Z.-C. Gu, K. Sun, and L. Sheng, Nature Commun. **2**, 389 (2011).
 - ⁷ Y.-F. Wang, Z.-C. Gu, C.-D. Gong, and D. N. Sheng, Phys. Rev. Lett. **107**, 146803 (2011).
 - ⁸ N. Regnault and B. A. Bernevig, Phys. Rev. X **1**, 021014 (2011).
 - ⁹ B. A. Bernevig and N. Regnault, Phys. Rev. B **85**, 075128 (2012).
 - ¹⁰ Y.-F. Wang, H. Yao, Z.-C. Gu, C.-D. Gong, and D. N. Sheng, Phys. Rev. Lett. **108**, 126805 (2012).
 - ¹¹ S. Yang, K. Sun, and S. Das Sarma, Phys. Rev. B **85**, 205124 (2012).
 - ¹² S. Yang, Z.-C. Gu, K. Sun, and S. Das Sarma, ArXiv e-prints: 1205.5792 (2012), arXiv:1205.5792 [cond-mat.str-el].
 - ¹³ S. A. Parameswaran, R. Roy, and S. L. Sondhi, Phys. Rev. B **85**, 241308 (2012).
 - ¹⁴ C. Lee, R. Thomale, and X. Qi, e-print arXiv:1207.5587 (2012).
 - ¹⁵ X.-L. Qi, Phys. Rev. Lett. **107**, 126803 (2011).
 - ¹⁶ M. Barkeshli and X.-L. Qi, Phys. Rev. X **2**, 031013 (2012).
 - ¹⁷ A. Sterdyniak, C. Repellin, B. A. Bernevig, and N. Regnault, arXiv preprint arXiv:1207.6385 (2012).
 - ¹⁸ Y.-F. Wang, H. Yao, C.-D. Gong, and D. N. Sheng, Phys. Rev. B **86**, 201101 (2012).
 - ¹⁹ Z. Liu, E. J. Bergholtz, H. Fan, and A. M. Läuchli, Physical Review Letters **109**, 186805 (2012).
 - ²⁰ Y.-L. Wu, N. Regnault, and B. A. Bernevig, ArXiv e-prints (2012), arXiv:1210.6356 [cond-mat.str-el].
 - ²¹ Y.-L. Wu, N. Regnault, and B. A. Bernevig, Phys. Rev. B **86**, 085129 (2012).
 - ²² T. Scaffidi and G. Möller, Phys. Rev. Lett. **109**, 246805 (2012).
 - ²³ Z. Liu and E. J. Bergholtz, Phys. Rev. B **87**, 035306 (2013).
 - ²⁴ X.-L. Qi, Y.-S. Wu, and S.-C. Zhang, Phys. Rev. B **74**, 085308 (2006).
 - ²⁵ C. Fang, M. J. Gilbert, and B. A. Bernevig, ArXiv e-prints (2012), arXiv:1208.4603 [cond-mat.mes-hall].